

SELF-INTERSECTIONS OF SOME IMMERSED MANIFOLDS

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The purpose of this paper is to prove a theorem, stated in §3, which describes the self-intersection manifolds of some immersions of differentiable manifolds in euclidean space in the metastable range. The description is in terms of Z_2 -bordism, a concept introduced by Conner and Floyd [2] generalizing the cobordism theory of Thom [15]. In §4 some consequences of the theorem are stated and in §5 a method is developed for producing immersions of spheres whose self-intersections are the real projective spaces P^n for all n .

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1. All manifolds and maps will be differentiable of class C^∞ , and manifolds will be compact and without boundary unless otherwise stated. All homology will have Z_2 coefficients unless otherwise stated.

In addition to the standard results of differential topology, as found in Lang [5] or Milnor [9]–[12], some special material will be needed.

Two compact manifolds N_0^n, N_1^n are (unoriented) cobordant if there exists a compact manifold with boundary W^{n+1} such that $\partial W = N_0 \cup N_1$. This can be shown to be an equivalence relation. The class of a manifold M will be written $[M]$.

The disjoint union operation $+$ and the cartesian product operation \times make the set of all cobordism classes into a graded commutative Z_2 -algebra \mathfrak{R} . The additive inverse of a cobordism class is itself, since $\partial(M \times I) = M \cup M$. The class of the empty manifold is the zero element. Thom [15] showed that \mathfrak{R} is a polynomial algebra with one generator in each dimension not of the form $2^j - 1$. The cobordism class of a manifold N in \mathfrak{R} is determined as follows: Let w_i be the i th Stiefel-Whitney class of the tangent bundle of N [11] and $[N]$ its fundamental homology class. Let π be a sequence (i_1, \dots, i_n) such that $\sum_j j i_j = n$. The π th *Stiefel-Whitney number* of N is the value of the index $\langle w_1^{i_1} w_2^{i_2} \cdots w_n^{i_n}, [N] \rangle$. Then two manifolds are cobordant iff these numbers are the same for all π .

These ideas received the following extension from Conner and Floyd [2]: The objects considered are manifolds with fixed-point free involutions, that is, diffeomorphisms T such that $T^2 = I$ and $Tx \neq x$ for all x . These will be written (M, T) .

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Two such, (M_0^m, T_0) , (M_1^m, T_1) , are cobordant if there exists (W^{m+1}, T) such that $\partial(W, T) = (M_0, T_0) \cup (M_1, T_1)$, the union of the given manifolds with their given involutions. Note that the quotient manifolds M_0/T_0 and M_1/T_1 are then cobordant in the original sense. Disjoint union, as above, equips the objects under this relation with the structure of a graded vector space over Z_2 , denoted $\mathfrak{R}(B_{Z_2})$. Further, $\mathfrak{R}(B_{Z_2})$ is a module over \mathfrak{R} . If $[N] \in \mathfrak{R}$ and $[(M, T)] \in \mathfrak{R}(B_{Z_2})$, then $(N \times M, 1 \times T)$ represents $[N][(M, T)]$, the involution $1 \times T$ acting in coordinate-wise fashion.

Conner and Floyd have shown that $\mathfrak{R}(B_{Z_2})$ is a free module over \mathfrak{R} with one generator in each dimension ≥ 0 . These may be taken to be (S^n, A^n) , the standard sphere furnished with the antipodal map. In other words, any element may be expressed as a sum $[(M, T)] = \sum_{i=0}^m [V_i][(S^i, A^i)]$, where V_i is of dimension $m-i$.

These equivariant cobordism classes are determined by *involution numbers* analogous to the Stiefel-Whitney numbers described above. Given (M^m, T) , there is in $H^1(M/T)$ an element c , the characteristic class of the double covering $M \rightarrow M/T$. An involution number of (M, T) is an integer (mod 2) $\langle w_1^{i_1} \cdots w_m^{i_m} c^r, [M/T] \rangle$, where $\sum_j j i_j + r = m$, and the w_j are the tangent Stiefel-Whitney classes of M/T . As above, two manifolds with involution are equivariantly cobordant iff all their involution numbers agree.

2. The immersions considered here will be of compact n -dimensional manifolds N^n in m -dimensional euclidean space R^m with the restriction of *complete regularity*.

DEFINITION. *A collection of subspaces of a vector space intersect transversely if their orthogonal complements are linearly independent.*

Let T_x denote the tangent space at a point x in a manifold.

DEFINITION. *An immersion $f: N^n \rightarrow R^m$ is completely regular if for x_1, \dots, x_r such that $f(x_1) = \cdots = f(x_r)$, $f_*T_{x_1}, \dots, f_*T_{x_r}$ intersect transversely.*

If $3n < 2m$, the case we will be interested in, complete regularity insures that f has no triple points; and it follows that the set $\bar{I}_f = \{x \in N \mid \exists y, f(x) = f(y)\}$ is a compact $(2n-m)$ -dimensional submanifold. Then the image $I_f = f\bar{I}_f$ is a submanifold of R^m , which is doubly covered by \bar{I}_f [7].

Let $f^2 = f \times f | N \times N - \Delta(N): N \times N - \Delta(N) \rightarrow R^m \times R^m$, where Δ denotes the diagonal. Then \bar{I}_f may be realized as $(f^2)^{-1} \Delta(R^m)$. The covering transformation T_{I_f} of $\bar{I}_f \rightarrow I_f$ is given by the transposition $(x, y) \rightarrow (y, x)$.

Completely regular immersions possess two important properties. They are *generic* in the sense of [16]: Any immersion of a compact manifold and its derivatives up to any order may be approximated arbitrarily closely by a regularly homotopic completely regular immersion, and any immersion close enough to a completely regular immersion g , whose first two derivatives approximate those of g , is completely regular. Secondly, they are *stable*, in that under a sufficiently small regular homotopy the diffeomorphism type of the immersion, in a suitably defined sense, does not change. In particular, the intersection manifolds are diffeomorphic [8].

PROPOSITION. *If f and g are regularly homotopic completely regular immersions $N^n \rightarrow R^m$, $3n < 2m$, then (\bar{I}_f, T_{I_f}) and (\bar{I}_g, T_{I_g}) are cobordant.*

According to Hirsch [4] there is an immersion $G: N \times I \rightarrow R^m \times I$ which is equal to $f \times \text{id}$ on $N \times [0, 1/3]$ and $g \times \text{id}$ on $N \times [2/3, 1]$. Now $f \times \text{id}$ and $g \times \text{id}$ are completely regular; for if, say, $f \times \text{id}(x, t_0) = (f(x), t_0) = (f(y), t_0) = f \times \text{id}(y, t_0)$, the orthogonal complements to the differentials of the tangent spaces in $R^m \times I$ are just the inclusions of those in R^m , which were assumed to be linearly independent.

Approximate G by a completely regular immersion H relative to $N \times [0, 1/3]$ and $N \times [2/3, 1]$. A calculation shows that H has no triple points, so (\bar{I}_H, T_{I_H}) is the required cobordism.

3. The *intersection number* of a completely regular immersion $f: N \rightarrow R^{2n}$ is the number (mod 2) of point self-intersections. Whitney [19] showed that for all n there exist immersions with intersection number taking either value.

A manifold N is said to be stably k -parallelizable, or briefly, $k\pi$, if for every complex K of dimension $\leq k$ and map $h: K \rightarrow N$, $k^*\nu$ is trivial, where ν is the stable normal bundle of N . This is equivalent to the condition that for some cell decomposition of N the stable normal bundle restricted to the k -skeleton is trivial. If $k \geq n$, then N is called a π -manifold.

Given a formal power series $\alpha = \sum_i \alpha_i x^i$ over Z_2 . It is known that α is a unit iff $\alpha_0 = 1$, and in that case its inverse may be formed by rational operations. Let w_N denote the total tangent Stiefel-Whitney class of a manifold N .

THEOREM 1. *Let N^n be a $k\pi$ -manifold, $k \geq 2n - m$, and $f: N \rightarrow R^m$ be a completely regular immersion with $3n < 2m$. Let jf be the composition $N \xrightarrow{f} R^m \xrightarrow{j} R^{2n}$, and c be the characteristic class of the covering $\bar{I}_f \rightarrow I_f$. Then $w_{I_f} = (1 + c)^{n-m} = 1/(1 + c)^{m-n}$. If the intersection number of jf is 0, then $c^{2n-m} = 0$ and \bar{I}_f equivariantly bounds. If the intersection number of jf is 1, then $c^{2n-m} \neq 0$ and \bar{I}_f is equivariantly nonbounding.*

Thus the cobordism type of I_f and the equivariant cobordism type of \bar{I}_f may be computed, as explained in §1, and depend only on m, n , and the intersection number of jf .

An analogue of this theorem can be proved by the same methods with the weaker hypothesis $4n < 3m$, but the author prefers to avoid the necessary complications of statement and proof. Note that in the expansion $[\bar{I}_f] = \sum_{i=0}^{2n-m} [M_i][S^i, A^i]$ the only term which can have nonzero involution number $\langle c^{2n-m}, [M_i \times S^i/A^i] \rangle$ is $[S^{2n-m}, A^{2n-m}]$, since $M_i \times S^i/A^i = M_i \times P^i$, where P^i is real projective i -space. Thus, if \bar{I}_f does not equivariantly bound, the coefficient of $[S^{2n-m}, A^{2n-m}]$ is 1. I have no examples of immersions for which any other terms are not zero, but nothing seems to preclude the possibility of their existence.

COROLLARY 1. *If $[\bar{I}_f] \neq 0$, then $[\bar{I}_f] = [S^{2n-m}, A^{2n-m}]$ iff $m > 2n - 2^s$, where 2^s is the greatest power of 2 dividing $n + 1$.*

This follows from comparing $w_{p^{2n-m}} = (1+c)^{2n-m+1}$ and $(1+c)^{n-m}$.

COROLLARY 2. *If n is even, then $[\bar{I}_f] = 0$ for every completely regular immersion $f: N \rightarrow R^m$ with $m < 2n$.*

Smale and Lashof [6] have shown that if in this case the intersection number of jf is not zero, then the Euler class $\chi \in H^n(N; Z)$ is not zero. But Hirsch [4] has shown that $\chi = 0$ is a necessary and sufficient condition for an immersion of N in R^{2n} to be regularly homotopic to one in R^{2n-1} .

The proof of the theorem will be preceded by several lemmas. Let $k: R^m \rightarrow R^{m+1}$ be the standard inclusion.

LEMMA 1. *Let $f: N^n \rightarrow R^m$ be a completely regular immersion, $3n < 2m$, with \bar{I}_f representing $\sum_{i=0}^{2n-m} [M_i][S^i, A^i]$. Then for any completely regular immersion g regularly homotopic to kf , \bar{I}_g represents $\sum_{i=1}^{2n-m} [M_i][S^{i-1}, A^{i-1}]$.*

By the proposition it will suffice to find one such g . The proof will be divided into two parts.

Given a manifold with involution (V^v, T) , the line bundle E_v associated to T has total space the quotient of $V \times R$ by the relation $(x, s) \sim (Tx, -s)$, base space V/T , and projection $p_v: (x, s) \rightarrow [x]$, where $[x]$ is the class of x in V/T .

(1) Let V^v be a manifold with involution T , representing $\sum_{i=0}^v [M_i][S^i, A^i] \in \mathfrak{R}(B_{Z_2})$. Then if $r: V/T \rightarrow E_v$ is a section which is transverse to the zero section and 0_r is the set of zeros, then $(P_v^{-1}0_r, T | P_v^{-1}0_r)$ represents $\sum_{i=1}^v [M_i][S^{i-1}, A^{i-1}]$.

By the hypotheses there exists an equivariant cobordism (W, R) between (V, T) and $\bigcup_{i=1}^v (M_i \times S^i, 1 \times A^i) = (X, A)$. I claim that there is a section q_i of $E_{M_i} \times S^i$, transverse to the zero section and such that 0_{q_i} is $M_i \times S^{i-1}/T = M_i \times P^{i-1}$. Consider S^i embedded as the unit sphere in R^{i+1} and define $h(x)$, $x \in S^i$, to be the $(i+1)$ th coordinate x_{i+1} of x . Then define $\bar{q}_i(y, x) = (y, x, h(x)) \in M_i \times S^i \times R$. It is evident that $h(A^i x) = h(-x) = -h(x)$. Thus $\bar{q}_i(x)$ respects the identifications made in defining $E_{M_i} \times S^i$ and passes to a section $q_i: M_i \times S^i/A^i = M_i \times P^i \rightarrow E_{M_i} \times S^i$. Since $\partial h/\partial x_{i+1}$ is $\neq 0$ on S^{i-1} , \bar{q}_i is transverse to the zero section and thus so is q_i . Define $q: X \rightarrow E_X$ to be the union of the q_i . Then $0_q = \bigcup_{i=1}^n M_i \times P^{i-1}$.

X/A and V/T have neighborhoods diffeomorphic to the products $(X/A) \times I$ and $(V/T) \times I$ in W/R . Define a section $v: W/R \rightarrow E_W$ to be the products $q \times \text{id}$ and $r \times \text{id}$ in these neighborhoods and transverse to the zero section elsewhere. Then 0_v is a manifold with boundary $0_r \cup 0_q$. $(p_w^{-1}0_v, R | p_w^{-1}0_v)$ is a manifold with involution, restricting on its boundary to $(p_v^{-1}0_r, T | p_v^{-1}0_r) \cup (p_x^{-1}0_q, A | p_x^{-1}0_q)$. But

$$\begin{aligned} (p_x^{-1}0_q, A | p_x^{-1}0_q) &= \bigcup_{i=1}^v (M_i \times S^{i-1}, 1 \times A^i | M_i \times S^{i-1}) \\ &= \bigcup_{i=1}^v (M_i \times S^{i-1}, 1 \times A^{i-1}), \end{aligned}$$

which establishes (1).

(2) Let f be an immersion as above. Then if r is any section of E_{I_f} transverse to the zero section there is a completely regular immersion g regularly homotopic to kf such that $\bar{I}_g = (p_{I_f}^{-1}0_r, T_{I_f} \mid p_{I_f}^{-1}0_r)$.

g will be constructed by first defining a map $m: \bar{I}_f \rightarrow I_f \times R \subset R^{m+1}$ and then extending m to an immersion of N .

By the definition of E_{I_f} there is a bundle map $s: \bar{I}_f \times R \rightarrow E_{I_f}$. Then there is the diagram

$$\begin{array}{ccc} & R & \\ I \uparrow & & \\ \bar{I}_f \times R & \xrightarrow{s} & E_{I_f} \\ \downarrow & & \downarrow p_f \\ \bar{I}_f & \xrightarrow{\pi} & I_f \end{array}$$

Let \bar{r} be the section of \bar{I}_f induced from r by π . That is, $\bar{r}(x) = s^{-1}r\pi(x)$. Define $m(x) = (f(x), \bar{r}(x))$. Then since $\bar{r}Tx = -\bar{r}T_x$, by the identifications made in defining E_{I_f} , there exists y such that $m(x) = m(y)$ iff $x \in \pi^{-1}(0_r)$. Further, $m(\bar{I}_f)$ meets I_f transversely in $I_f \times R$, since r meets the zero section in E_{I_f} transversely and the bundle map s is locally an isomorphism.

Let U be a tubular ε -neighborhood of \bar{I}_f in N for small $\varepsilon > 0$.

Let $d(x)$ be the distance of x from \bar{I}_f in N . Let ρ be a real valued C^∞ function defined on the reals satisfying

- (i) $\rho(y) = 1, y \leq 0; \rho(y) = 0, y \geq \varepsilon$, and
- (ii) $d\rho/dx \leq 0$ [14].

Define g as follows: If $x \notin U$, $g(x) = kf(x)$. If $w \in \bar{I}_f$ and x is in the fiber of w in U , U_w , $g(x) = [f(x), \rho(d(x)) \cdot \bar{r}(w)] \in R^m \times R = R^{m+1}$. Then g extends m , and its self-intersections are precisely those of m . Define a homotopy H between kf and g by $H(x, t) = [f(x), 0]$ if $x \notin U$, and if $x \in U_w$, $H(x, t) = [f(x), \rho(d(x)) \cdot \bar{r}(w) \cdot t]$.

Then g is an immersion. This is clear if $x \notin U$. If $x \in U_w$, we may select a neighborhood of w in \bar{I}_f on which m is an embedding. Then it is clear that g embeds the restriction of N to this neighborhood. Similarly $H(N, t)$ is an immersion for all t , so H is a regular homotopy. Finally we must show that at any intersection point $z = f(x) = f(y)$ of g the intersection is transverse.

In the case of an immersion like g which has no triple points, this is easily seen to be equivalent to the condition that the images of the tangent spaces at x and y span that at z .

Now z is also an intersection point of f . The projection $R^{m+1} = R^m \times R \rightarrow R^m$ carries g into f , which was assumed to be completely regular, so the tangent spaces of the two leaves at z span the subspace of the tangent space of R^{m+1} orthogonal to the last coordinate. Since g extends m and the two leaves of $m(\bar{I}_f)$ at each intersection were shown to span the tangent space to $I_f \times R$, where R is the last coordinate, the two tangent spaces at z span the tangent space to R^{m+1} .

Combining (1) and (2) yields the lemma.

COROLLARY. *If f is an immersion as above and $\bar{I}_f = (S^{2n-m}, A^{2n-m})$, then we may choose g so that $\bar{I}_g = (S^{2n-m-1}, A^{2n-m-1})$.*

In this case the construction of a section q over X/A made in (1) may be applied directly on I_f . (2) needs no modification.

LEMMA 2. *Let N^n be a $k\pi$ -manifold, $k \geq 2n-m$, and $f: N \rightarrow R^m$ a completely regular immersion with $3n < 2m$. Let α be the line bundle over I_f associated to $\pi: \bar{I}_f \rightarrow I_f$, ϵ the trivial line bundle, and c the characteristic class of α . Then the normal bundle ν of I_f in R^m is $(m-n)\alpha \oplus (m-n)\epsilon$ and thus its total Stiefel-Whitney class is $(1+c)^{m-n}$.*

I claim that \bar{I}_f is a π -manifold: it is diffeomorphic to $(f^2)^{-1} \Delta(R^m) \subset N \times M$. The normal bundle to $(f^2)^{-1} \Delta(R^m)$ in $N \times N$ is trivial, being induced from that of $\Delta(R^m)$ in $R^m \times R^m$. Now \bar{I}_f is of dimension $2n-m$, and being a differentiable manifold is homeomorphic to a complex [13]. Since $k \geq 2n-m$, it will suffice to show that $N \times N$ is a $k\pi$ -manifold. Let p_1, p_2 be the projections of $N \times N$ on its factors. If η is the stable normal bundle of N , the stable normal bundle of $N \times N$ may be written $p_1^* \eta \oplus p_2^* \eta$. If N^r is the r -skeleton of N in some cell decomposition, then under p_1 , $N^k \times N$ maps onto N^k , so $p_1^* \eta|_{N^k \times N}$ is trivial. Similarly $p_2^* \eta|_{N \times N^k}$ is trivial. Thus $p_1^* \eta \oplus p_2^* \eta|_{N^k \times N^k}$ is trivial. But $(N \times N)^k \subset N^k \times N^k$.

\bar{I}_f is contained in N with normal bundle ξ of dimension $n - (2n-m) = m-n$. Since N is a $k\pi$ -manifold and $k \geq 2n-m$, ξ is stably trivial. Since the fiber dimension $m-n$ is greater than the base dimension $2n-m$, ξ is trivial. Let $\{r_i\}$, $i=1, \dots, m-n$, be a framing of ξ . If $f(x)=f(y)=z$, the normal space to I_f at z is the sum of the normal spaces to \bar{I}_f in N at x and y , since the intersection is transverse. Each r_i thus determines a two-dimensional subbundle ν_i of ν . The ν_i are mutually isomorphic, for an automorphism of ξ taking r_i to r_j induces an automorphism of ν taking ν_i to ν_j .

Consider the trivial 2-plane bundle θ over \bar{I}_f spanned by r_1 and a second vector field s_1 . Define a bundle map $l: \theta \rightarrow \nu_1$ by $lr_1(x) = f_* r_1(x)$, $ls_1(x) = f_* r_1(Tx)$. The identifications made under this map are $ar_1(x) + bs_1(x) \sim as_1(Tx) + br_1(Tx)$. Let $q_1 = r_1 + s_1$, $q_2 = r_1 - s_1$. Then in terms of the $\{q_i\}$ the identifications are $cq_1(x) + dq_2(x) \sim cq_1(Tx) - dq_2(Tx)$, so the image of the span of q_1 is trivial, and the image of that of q_2 is the line bundle associated to the covering $\bar{I}_f \rightarrow I_f$.

Proof of Theorem 1. It follows from Lemma 2 that the normal Stiefel-Whitney class of I_f is $(1+c)^{m-n}$, where c is truncated to some height $\leq 2n-m$. Thus by Whitney duality the tangent class is $1/(1+c)^{m-n}$. Suppose that the intersection number of jj is 0. Then $c^{2m-n} = 0$, for if not \bar{I}_f has the nonzero involution number $\langle c^{2m-n}, [I_f] \rangle$, and since $[S^{2n-m}, A^{2n-m}]$ is the only term with this involution number, it must have coefficient 1 in \bar{I}_f . By descending induction using Lemma 1 the intersection number of jj is 1, a contradiction. Then \bar{I}_f has no nonzero involution numbers and is an equivariant boundary.

If the intersection number of jf is 1 we see using Lemma 1 that the coefficient of $[S^{2n-m}, A^{2n-m}]$ is 1, and so $\langle c^{2n-m}, [I_r] \rangle \neq 0$, and \tilde{I}_r does not equivariantly bound.

4. Robert Wells [17] has shown that if $3n < 2m$, $k \geq 2n - m$, and N^n is a $k\pi$ -manifold, then every manifold-with-involution occurring as the self-intersection of a completely regular immersion $N \rightarrow R^m$ may be realized as that of some completely regular immersion of the sphere $S^n \rightarrow R^m$. Thus according to Theorem 1 the existence of immersions of $k\pi$ -manifolds in the above range with equivariantly nonbounding self-intersections is equivalent to the existence of immersions $S^n \rightarrow R^m$ which go under the inclusion $R^m \rightarrow R^{2n}$ into immersions with intersection number 1 (mod 2).

Recall Smale's theory of immersions of spheres in euclidean space [14]. If $n < m$, the group of immersions of S^n in R^m under the operation of connected sum is isomorphic to $\pi_n(V_{m,n})$, where $V_{m,n}$ is the Stiefel manifold of orthogonal n -frames in m -space. Further, the inclusion of immersions $S^n \rightarrow R^m \rightarrow R^{m+1}$ corresponds to the homomorphism $\pi_n(V_{m,n}) \rightarrow \pi_n(V_{m+1,n})$ induced from the map $V_{m,n} \rightarrow V_{m+1,n}$. It was remarked in §3 that self-intersections are uninteresting for n even. We thus confine attention to odd n . In this case Smale's construction shows that the immersion of S^n in R^{2n} with intersection number 1 corresponds to the nonzero element of $\pi_n(V_{2n,n}) \approx Z_2$. We thus have

THEOREM 2. *Consider n, m with $3n < 2m$ and n odd. Then there exists an n -dimensional $k\pi$ -manifold, $k \geq 2n - m$, completely regularly immersed in R^m with equivariantly nonbounding self-intersection iff the homomorphism $\pi_n(V_{m,n}) \rightarrow \pi_n(V_{2n,n})$ is not zero. In fact the manifold may always be taken to be a sphere.*

5. In many cases the question of the existence of self-intersections may be settled geometrically, answering the homotopy-theoretic question of the last section indirectly.

Throughout this section P^r will denote real projective space and S^r the sphere considered as a covering space of P^r . I will consider immersions $f: P^n \rightarrow R^m$ with $3n + 1 < 2m$. The method used here to find examples of self-intersections consists essentially of considering maps of projective spaces, passing to the associated maps of spheres given by the covering projection, and putting these into general position.

These examples are special cases of those found independently by Robert Wells [17]. Wells has also contributed to the simplification of my hypotheses.

Let α_r be the canonical line bundle over P^r , and c_r the order of α_r in $\tilde{K}\tilde{O}(P^r)$ (the reduced KO group).

THEOREM 3. *Given n and m , $3n + 1 < 2m$. Then if $n + 1 \not\equiv 0 \pmod{c_{2n-m}}$ there exists a completely regular immersion $S^n \rightarrow R^{m+1}$ with self-intersection P^{2n-m-1} .*

Since by Adams' calculations [1] c_r is finite, such immersions exist for all values of $2n - m$ for an infinite number of m .

The principal fact needed is as follows:

LEMMA 3. *Given n and m , $3n < 2m$ such that the stable normal bundle of P^n restricted to P^{2n-m} is trivial. Then there is a map $g: S^n \rightarrow R^m$ whose singular set in S^n has two components with the following structure: One component is a $(2n-m)$ -sphere with image P^{2n-m} , of which a neighborhood is mapped by a completely regular immersion. The other is a $(2n-m)$ -manifold containing the singular points.*

Consider any map $P^n \rightarrow R^m$. Haefliger [3] shows that in the stated dimensions it may be approximated by a map f whose singular set, Y_f , is a submanifold of dimension $2n-m$.

Since $2n-m+2n-m < n$ there is by general position an isotopy of P^{2n-m} in P^n separating P^{2n-m} from Y_f . By the isotopy extension theorem [12], this isotopy extends to an isotopy $H: P^n \times I \rightarrow P^n$ such that for $fH_1: P^n \rightarrow R^m$, $fH_1(P^{2n-m}) \cap Y_{fH_1} = \emptyset$. I will assume this to be the case, writing f for fH_1 .

Let $P^n - Y_f$ have the Riemannian metric d induced from that of R^m by f and let K be a tubular ε -neighborhood of P^{2n-m} in P^n , with ε chosen so that $K \cap Y_f = \emptyset$. Then since if in the given range the normal bundle of P^n restricted to P^{2n-m} is stably trivial it is trivial, and K contracts onto P^{2n-m} , we may choose a framing $\{r_i\}$, $i=0, \dots, m-n-1$, of the normal bundle of P^n restricted to K .

P^r is covered by $r+1$ coordinate neighborhoods $\{V_i\}$, where V_i is the region where the i th homogeneous coordinate is nonzero. V_i lifts to S^r as the union of the two open hemispheres U_i^+ and U_i^- . Let $\delta_i(x) = 1$ if $x \in U_i^+$, $= -1$ if $x \in U_i^-$, and $= 0$ if $x \notin U_i^+$ or U_i^- .

$P^{2n-m} \subset P^n$ may be considered as the intersection of $m-n$ copies of P^{n-1} , $\{P_i^{n-1}\}$, where P_i^{n-1} is the set of zeros of the i th homogeneous coordinate.

Consider the normal disc bundle to P^n as a submanifold of R^m . Let $p: S^n \rightarrow P^n$ be the projection. Define a map $g': p^{-1}K \rightarrow R^m$ by

$$g'(x) = \sum_{i=0}^{m-n-1} \delta_i(x) d(px, P_i^{n-1}) r_i(fp x).$$

Let $g: S^n \rightarrow R^m$ be a map agreeing with g' on $p^{-1}K$ and whose singular set is a $(2n-m)$ -dimensional manifold, which exists by the theorem of Haefliger cited above.

I will show that $g|p^{-1}K$ is a completely regular immersion with $I_g = P^{2n-m}$. To show that $g|p^{-1}K$ is an immersion it will suffice that $\delta_i(x) d(px, P_i^{n-1})$ be a C^∞ function of x , for then g will give a local section over fK , since p is a local diffeomorphism, and thus g will be locally an embedding.

If $x \in U_i^+$, $\delta_i(x) d(px, P_i^{n-1}) = d(px, P_i^{n-1})$. If $x \in U_i^-$, $\delta_i(x) d(px, P_i^{n-1}) = -d(px, P_i^{n-1})$, and $d(px, P_i^{n-1})$ is C^∞ on $p^{-1}K \cap U_i^\pm$. Thus there remains only the case $x \notin U_i^+$, U_i^- . Let $S_i^{n-1} = S^n - (U_i^+ \cup U_i^-)$. Since p is locally a diffeomorphism, in a small neighborhood X of x we may write $\delta_i(x) d(px, P_i^{n-1}) = \delta_i(x) \bar{d}(x, S_i^{n-1})$, where \bar{d} is the metric on Z induced from d by p . Let x_n be the normal coordinate of

S_i^{n-1} in S^n , the positive direction being into U_i^+ . Then locally $\delta_i(x) \vec{d}(x, S_i^{n-1}) = \pm x_n$.

If $x_1, x_2 \in p^{-1}x$, $x \in K - P^{2n-m}$, then $\delta_i(x_1) \neq \delta_i(x_2)$ for some i , $0 \leq i < m-n$. Thus the self-intersection set of $g|p^{-1}K$ is P^{2n-m} .

We must show that the intersection is transverse. To do this it will suffice to show that at every intersection point $x = fp(x_1) = fp(x_2)$ the leaves containing x_1 and x_2 have distinct derivatives with respect to $m-n$ independent directions normal to fP^{2n-m} in fP^n . Let W be a small neighborhood of x in fP^n contained in fK . Let W_1 and W_2 be the components of $p^{-1}f^{-1}W$ which contain x_1 and x_2 .

Define a partial coordinate system in W by letting w_i be the normal coordinate to fP_i^{n-1} in fP^n with positive direction chosen so that $w_i(fpy) > 0$ if $y \in W_1 \cap U_i^+$, and thus $w_i(fpy) < 0$ if $y \in W_2 \cap U_i^+$. Then the w_i are independent functions on W , since the P_i^{n-1} intersect transversely in P^n , and they lift under $p^{-1}f^{-1}$ to give partial coordinate systems $\{w_i^1\}$, $\{w_i^2\}$. Let $g_i = g|w_i$. Since pf is a diffeomorphism on each of W_1 , W_2 , we may write g_1 and g_2 in terms of the coordinates $\{w_i\}$ instead of $\{w_i^1\}$, $\{w_i^2\}$. Then

$$\left[\frac{\partial g_j}{\partial w_i} \right]_x = \left[\frac{\partial}{\partial w_i} \right]_x \sum_{k=0}^{m-n-1} \delta_k(x_j) d(fp x_j, fP_i^{n-1}) r_k(fp x_j).$$

Rewriting as above, this is equal to

$$\left[\frac{\partial}{\partial w_i} \right]_x (-1)^{j+1} \sum_{k=0}^{m-n-1} w_k(fp x_j) r_k(fp x_j) = (-1)^{j+1} r_i(x).$$

Since $r_i(x) \neq 0$, $\partial g_1 / \partial w_i \neq \partial g_2 / \partial w_i$ for $0 \leq i < m-n-1$, and the intersection is transverse.

Proof of Theorem 3. The normal bundle of P^{2n-m} splits as the sum of the normal bundle of P^{2n-m} in P^n and the stable normal bundle of P^n restricted to P^{2n-m} . The former is the $(m-n)$ -fold sum of α . The tangent bundle of P^{2n-m} is stably the $(2n-m+1)$ -fold sum of α [10]. Thus, to apply Lemma 3 we must have that the stable normal bundle of P^{2n-m} is $(m-n)\alpha$. Since the stable normal bundle is uniquely characterized by the property that its sum with the stable tangent bundle is trivial, it will suffice that $(m-n) + (2n-m+1) = n+1 \cong 0 \pmod{c_{2n-m}}$. Then a map g exists as in Lemma 3. The following is easily established.

LEMMA 4. *Let A^a and B^b be disjoint submanifolds of S^n such that $a+b+1 < n$. Then A and B are separated by an $(n-1)$ -sphere embedded in S^n .*

Thus, since $2n-m+2n-m+1 < n$, $p^{-1}P^{2n-m}$ and $Y_g - p^{-1}P^{2n-m}$ may be separated by an $(n-1)$ -sphere in S^n , which is then embedded by g . So under the hypotheses of Lemma 3 there exists a disc D^n immersed in R^m with P^{2n-m} as self-intersection and which is embedded in a neighborhood V of ∂D^n .

Taking the composition $D^n \xrightarrow{g} R^m \rightarrow R^{m+1}$ we may attach a cone C over ∂D^n , forming a sphere S^n which is (topologically) embedded on $C \cup V$. Now by a

theorem of Haefliger [3], since $3n < 2m$, this embedding may be approximated by a differentiable embedding of $C \cup V$, leaving g unaltered outside of V .

Now we may apply the corollary to Lemma 1 to produce an immersion $S^n \rightarrow R^{m+1}$ with the desired self-intersection.

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